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ASYMPTOTIC METHODS FOR RELAXATION OSCILLATIONS

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Asymptotic methods for relaxation oscillations *)

by

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ABSTRACT

In this paper the characteristics of autonomous relaxation oscillations are discussed. Furthermore, singular perturbation techniques are used to analyse a Van der Pol equation with sinusoidal forcing term for large values of its parameter. Finally, this paper summarizes results on an asymptotic method in two small parameters for systems of coupled relaxation oscillations.

KEY WORDS & PHRASES: *Van der Pol equation, relaxation oscillation, frequency entrainment, asymptotic approximation*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

The subject we deal with is presented in three parts. First we discuss the characteristics of *autonomous relaxation oscillations*. Examples and a survey of mathematical methods for those examples will be given. As a prototype of a relaxation oscillation is commonly used the periodic solution of the Van der Pol equation for large values of its parameter. This equation will be discussed in more detail.

In the second part discontinuous approximations of periodic solutions of the nonautonomous Van der Pol equation,

$$(1.1) \qquad \frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = b\cos t, \qquad \nu \gg 1,$$

will be made. For b independent of ν this problem exhibits subharmonic oscillations of period $2\pi m$, where m is an integer of order $O(\nu)$. The construction of the approximation brings about certain conditions for b and ν. In the b,ν-plane overlapping regions are found where these conditions are satisfied. In the domain of overlap two periodic solutions with different periods are possible which is in agreement with analytical and numerical results. The case m odd was analyzed in [9], here we will give a modified method covering the case m odd as well as m even.

Finally, we will investigate *mutually coupled relaxation oscillations* of Van der Pol's type. Apart from the large parameter ν a second, small parameter related to the weakness of the coupling is introduced. Applying asymptotic methods in both parameters we can approximate periodic solutions of the coupled system. The results for this class of problems may help us to understand interesting phenomena occuring in systems of interacting biologic oscillators. We mention certain forms of frequency entrainment leading to wave phenomena in systems of spatially distributed oscillators.

2. AUTONOMOUS RELAXATION OSCILLATIONS

In 1926 Balthasar van der Pol wrote his paper "On relaxation oscillations" [17], in which the periodic solution of the differential equation

$$(2.1) \qquad \frac{d^2y}{dt^2} + \nu(y^2-1)\frac{dy}{dt} + y = 0$$

was investigated for different values of ν. Van der Pol remarked that the oscillations of (2.1) with $0 < \nu << 1$ differ considerably from those with $\nu >> 1$. In the first case the solution is a sinusoidal oscillation with a period close to 2π, while for ν large the solution changes in time alternately very slow and very fast with a period propertional to ν, see figure 2.1. For all $\nu > 0$ the oscillation has an amplitude $a(\nu)$ close to 2.
Van der Pol worked with a triode-circuit, in which fluctuations of the potential are described by (2.1). The parameter ν represents a time constant of the electrical system, the so-called time of relaxation. Since for $\nu >> 1$ the period is proportional to this parameter, he proposed to call the corresponding periodic solution a *relaxation oscillation.*

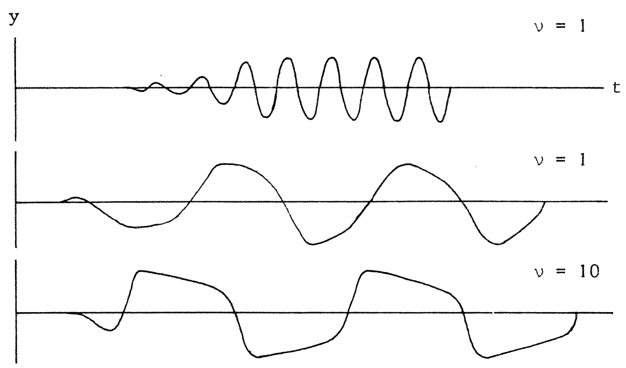

Fig. 2.1. Solutions of the Van der Pol equation for different values of the parameter.

In order to formulate a definition of relaxation oscillations for general autonomous systems of differential equations, we try to take some distance from Van der Pol's equation and introduce a new time-scale $\tau = t/\nu$ and a small parameter $\varepsilon = 1/\nu$. We consider the system of n equations

$$(2.2) \qquad p_i(\varepsilon)\,\frac{dx_i}{d\tau} = h_i(x_1,x_2,\ldots,x_n;\varepsilon), \qquad i = 1,2,\ldots,n,$$

where p_i and h_i are continuous functions in x and ε for $0 < \varepsilon \le \varepsilon_0$ with ε_0 sufficiently small. It is assumed that

$$(2.3) \qquad \lim_{\varepsilon\to 0} h_i(x,\varepsilon) \quad \text{exists for bounded } x,$$

$$(2.4) \qquad p_1,p_2,\ldots,p_k \to 0, \quad p_{k+1},\ldots,p_n \to 1 \text{ as } \varepsilon \to 0 \quad (1\le k\le n).$$

Let the system (2.2) have a periodic solution with period $T(\varepsilon)$ and with closed trajectory $C(\varepsilon)$ in $\mathbb{R}^n$ such that

$$(2.5) \qquad x \in C(\varepsilon) \text{ implies } |x| \text{ is uniformly bounded for } 0 < \varepsilon \le \varepsilon_0$$

$$(2.6) \qquad \lim_{\varepsilon\to 0} T(\varepsilon) \text{ exists and is nonzero.}$$

DEFINITION 2.1. A periodic solution of (2.2) with period $T(\varepsilon)$ and closed trajectory $C(\varepsilon)$ satisfying (2.3) - (2.6) is called a relaxation oscillation if the converging sequences $\{x_{\varepsilon_q}\}$, $x_{\varepsilon_q} \in C(\varepsilon_q)$ with $\varepsilon_q \to 0$ as $q \to \infty$ form two nonempty sets X_r and X_s:

$$(2.7) \qquad X_r = [\{x_{\varepsilon_q}\} \mid h_i(x_{\varepsilon_q},\varepsilon_q)/p_i(\varepsilon_q) \text{ converges as } q\to\infty \text{ for } i = 1,..,n],$$

$$(2.8) \qquad X_s = [\{x_{\varepsilon_q}\} \mid p_i(\varepsilon_q)/h_i(x_{\varepsilon_q},\varepsilon_q) \to 0 \text{ as } q \to \infty \text{ for some } i].$$

It is remarked that only nonlinear systems of the type (2.2) may exhibit relaxation oscillations. Furthermore, it is worth to mention that this definition does not provide a decisive answer on the stability of relaxation oscillations. At this point our definition does not concretize the existing vague idea that relaxation oscillations are asymptotically stable and, in case of forced oscillations, exhibit the phenomenon of frequency entrainment. There are Lyapunov stable (but not asymptotically stable) oscillations, which pass alternately the two characteristic phases of slow change and fast change in time as described by (2.7) and (2.8). It would lead to considerable confusion when these oscillations were termed differently. As an example of such oscillation we mention the periodic solutions of the

Volterra-Lotka equations for a certain range of the parameters. This system of equations has the form

$$\text{(2.9a)}\qquad \frac{dx_1^*}{dt^*} = x_1^*(-b+\beta x_2^*),$$

$$\text{(2.9b)}\qquad \frac{dx_2^*}{dt^*} = x_2^*(a-\alpha x_1^*).$$

Assuming that a << b we substitute

$$\text{(2.10ab)}\qquad x_1^* = \frac{b}{\alpha}\, x_1, \qquad t^* = \frac{1}{a}\, t,$$

$$\text{(2.10cd)}\qquad x_2^* = \frac{b}{\beta}\, x_2, \qquad a = \varepsilon b,$$

so that the system transforms into

$$\text{(2.11a)}\qquad \varepsilon\, \frac{dx_1}{dt} = x_1(-1+x_2),$$

$$\text{(2.11b)}\qquad \varepsilon\, \frac{dx_2}{dt} = x_2(\varepsilon - x_1), \qquad 0 < \varepsilon << 1.$$

A similar transformation can be made if a >> b. The system (2.11) has a one parameter family of periodic solutions with the equilibrium $(x_1,x_2) = (\varepsilon,1)$ as center point. In figure 2.2 we sketch the time-dependent behaviour of a periodic solution. In [8] it has been computed that the period satisfies

$$\text{(2.12)}\qquad T(\varepsilon) = (\mu-\theta) + \left(\frac{-1}{1-\theta} + \frac{1}{1-\mu}\right)\varepsilon\, \log\varepsilon \left[\frac{1}{1-\theta} - \frac{1}{1-\mu} + \frac{1}{1-\theta}\log\{(1-\theta)\log\frac{1}{\theta}\}\right.$$
$$\left. - \frac{1}{1-\mu}\log\{(\mu-1)\log\mu\} + I(\theta) + I(\mu)\right]\varepsilon + O(\varepsilon^2\log^2\varepsilon)$$

with

$$\theta - \log\theta = \mu - \log\mu = \sigma > 1, \qquad \theta < 1 < \mu,$$

$$I(\alpha) = \mathrm{sign}(1-\alpha)\int_0^{-\ln\alpha} \left\{\frac{1}{(x+\alpha(1-e^x)} - \frac{1}{(1-\alpha)x}\right\} dx,$$

where $\theta = x_{2min}$, $\mu = x_{2max}$, and $\sigma = x_{1max} + O(\varepsilon\log\varepsilon)$.

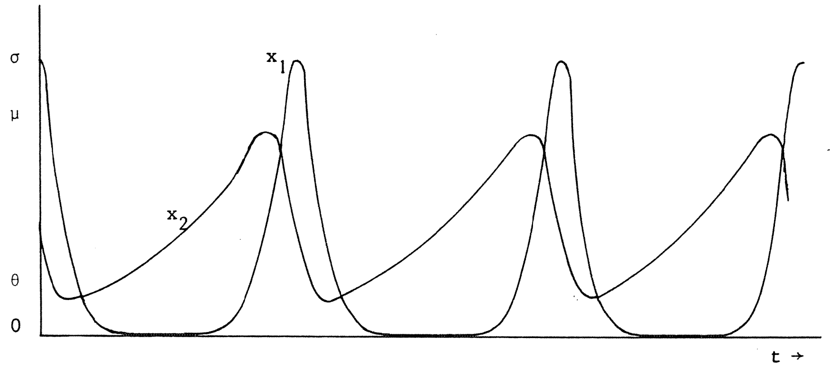

Fig. 2.2. A Volterra-Lotka system

For the proof of existence of periodic solutions of (2.2), mainly two types of methods are applied in literature. The first one is based on the *theorem of Poincaré-Bendixson*. For the Van der Pol equation and its generalizations this has been done a.o. by LEVINSON and SMITH [13], LASALLE [12] and PONZO and WAX [17]. This method only applies to second order autonomous systems. The other type of method uses *fixed point theorems*, see the work of CARTWRIGHT and LITTLEWOOD [4] for Van der Pol's equation with a forcing term and the theory of MISHCHENKO and PONTRYAGIN [16] for systems of the type (2.2).

An *asymptotic expansion* for the periodic solution of the Van der Pol equation with $\nu >> 1$ has been given by DORODNICYN [6]. Because of the changing behaviour of relaxations oscillations one has to apply methods different from those for almost linear oscillations as developed by BOGOLIUBOV and MITROPOLSKY [3]. Asymptotic methods for relaxation oscillations bear some resemblance with boundary layer techniques in fluid mechanics. One constructs local asymptotic approximations for each interval where the periodic solution has its own characteristic behaviour. The integration constants in these approximations are found by matching adjacent local approximations. In [2] the asymptotic method of Dorodnicyn has been modified at the point of matching. It was necessary to add a fifth local approximation in order to obtain a complete picture of the periodic solution of (2.1) for ν large. The following expressions for the amplitude and period were obtained.

(2.13) $$a(\nu) = 2 + \frac{1}{3}\alpha\nu^{-4/3} + (\frac{1}{3}b_1 - \frac{16}{27}\log\nu - \frac{1}{9} + \frac{2}{9}\log 2 - \frac{8}{9}\log 3)\nu^{-2} +$$

$$(\frac{1}{3}b_2 - \frac{2}{27}\alpha^2)\nu^{-8/3} + (\frac{1}{3}b_3 + \frac{104}{243}\alpha\log\nu - \frac{4}{27}\alpha b_1 - \frac{91}{486}\alpha + \frac{52}{81}\alpha\log 3 +$$

$$- \frac{13}{81}\alpha \log 2)\nu^{-10/3} + o(\nu^{-10/3}),$$

(2.14) $$T(\nu) = (3-2\log 2)\nu + 3\alpha\nu^{-1/3} - \frac{2}{3}\nu^{-1}\log\nu + \{\log 2 - \log 3 + 3b_1 +$$

$$- 1 - \log\pi - 2\log Ai'(-\alpha)\}\nu^{-1} + o(\nu^{-1}),$$

with

$$\alpha = 2.33811, \quad Ai'(-\alpha) = 0.70121,$$

$$b_1 = 0.17235, \quad b_2 = 0.61778, \quad b_3 = -0.55045.$$

3. THE VAN DER POL EQUATION WITH FORCING TERM

We study the Van der Pol equation with a periodic forcing term for large values of the parameter ν:

(3.1) $$\frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = b\cos t.$$

For $b = 0$ the periodic solution is an autonomous relaxation oscillation as described in the preceding section. For $b > 0$ the system may have a periodic solution with a period m times the period of the driving term; this phenomenon is called *subharmonic entrainment*. The conditions on the values of ν and b under which this synchronization phenomenon occurs are derived in this section as the result of a formal approximation of the periodic solution by singular perturbation techniques with $1/\nu$ acting as a small parameter. These conditions bound regions in the b,ν-plane where a solution with period $2\pi m$, might exist. It turns out that the regions, belonging to different values of m, have overlap; this is in agreement with results based on analytical-topological methods by LITTLEWOOD [14] and with numerical results by FLAHERTY and HOPPENSTEADT [7]. Making some modifications in the method of GRASMAN, VELING and WILLEMS [9] we will construct here a slightly different (lower order) approximation so that the case m even also can be included.

The synchronized solution of (3.1) with $b > 0$ can be considered as the sum the autonomous relaxation oscillation and a small harmonic oscillation. We shall make local approximations in different regions, see figure 3.1. To state the formal conditions for synchronization it is not necessary to consider more regions (as done in [9]). We just will use the knowledge that the jumps from ±1 to ∓2 take place in a time $o(1)$. The method we use is related to Cole's treatment of the autonomous equation, see [5]. In the regions A and $\bar{A}$ we use two time scales, while in the regions B and $\bar{B}$ a stretching procedure with respect to the dependent variable is applied.

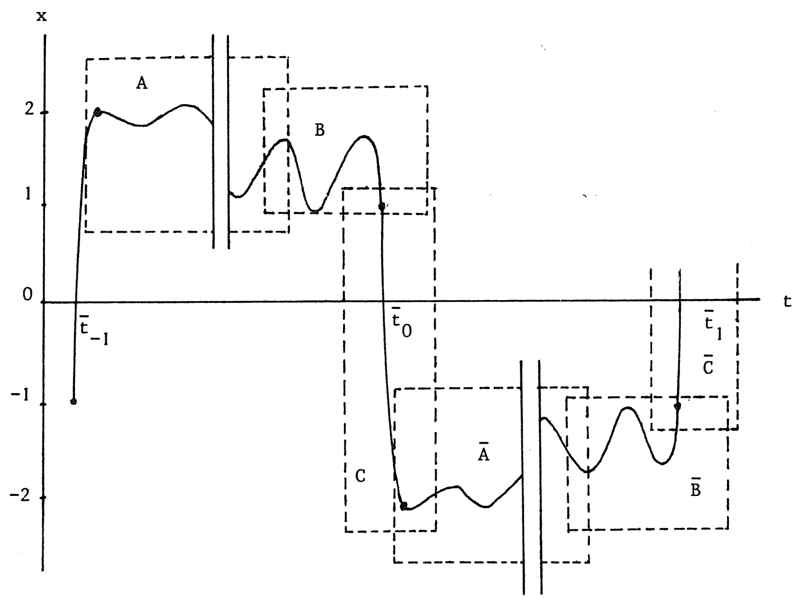

Fig. 3.1. Forced oscillations of the Van der Pol equation.

Region A

In this region the solution decays from the value 2 to 1 and has small amplitude oscillations of period 2π. We apply the two-variable expansion procedure by introducing a second independent variable

(3.2) $$\tau = (t-t_0)/\nu.$$

We suppose that the solution can be written as

(3.3) $$x = x_0(\tau) + x_1(t,\tau)\nu^{-1} + o(\nu^{-1}).$$

Substituting (3.3) into equation (3.1) and letting $\nu \to \infty$ we find the limit equation

(3.4) $$(x_0^2-1)\left(\frac{\partial x_1}{\partial t} + \frac{\partial x_0}{\partial \tau}\right) + x_0 = b \cos t.$$

Solving (3.4) with respect to x_1 we find

(3.5) $$x_1(t,\tau) = \frac{b \sin t}{x_0^2-1} - \frac{p_1(\tau)t}{x_0^2-1},$$

with

$$p_1(\tau) \equiv (x_0^2-1)\frac{\partial x_0}{\partial \tau} + x_0.$$

As seen from (3.5) the term with p_1 is secular in the t variable, so we have to set $p_1(\tau) = 0$. Integration with respect to τ gives

(3.6) $$\log x_0 - \tfrac{1}{2}(x_0^2-1) = \tau.$$

In [9] it is demonstrated that the integration constants of (3.5) and (3.6) can be taken zero; the constant t_0 of (3.2) already accounts for these contributions. Thus,

(3.7) $$x_1(t,\tau) = \frac{b \sin t}{x_0^2(\tau)-1}.$$

When t approaches t_0, the behaviour of x_0 and x_1 is

(3.8ab) $$x_0 \approx 1 + \sqrt{t_0-t}\,\nu^{-1/2}, \qquad x_1 \approx \frac{b \sin t}{2\sqrt{t_0-t}}\,\nu^{1/2},$$

so for $t \to t_0$ the constructed solution behaves singular and looses its validity.

Region B

Let us suppose that for values of $t = t_0 + O(1)$ the solution is of the type

(3.9) $$x = 1 + U_0(t)\nu^{-1/2} + o(\nu^{-1/2}).$$

Substituting (3.9) into (3.1) and letting $\nu \to \infty$ we obtain

(3.10) $$2U_0\frac{dU_0}{dt} + 1 = b \cos t,$$

so

(3.11) $$U_0(t) = \sqrt{b \sin t + t_0 - t + E_0}.$$

For $t_0 - t >> 1$, (3.12) behaves as

$$U_0(t) \approx \sqrt{t_0 - t} + \frac{b \sin t + E_0}{2\sqrt{t_0 - t}} . \tag{3.13}$$

By inspection (3.13) matches (3.8) if $E_0 = 0$. Next we determine the point where the solution intersects the line $x = 1$. With an accuracy of $o(\nu^{-1/2})$ this will be at $t = \bar{t}_0$ satisfying $U_0(\bar{t}_0) = 0$ or

$$t_0 - \bar{t}_0 = - b \sin \bar{t}_0 . \tag{3.14}$$

As we know from [9] at $t = \bar{t}_0$ the asymptotic solution jumps from 1 to the value -2 provided that

$$\cos \bar{t}_0 < 1/b. \tag{3.15}$$

Region $\bar{A}$

Similar to the asymptotic solution of region A, we expand the solution in region $\bar{A}$ as

$$x = y_0(\tau) + y_1(t,\tau)\nu^{-1} + o(\nu^{-1}), \qquad \tau = (t-t_0)/\nu \tag{3.16}$$

with

$$\log(-y_0) - \tfrac{1}{2}(y_0^2-1) = \tau + (t_0-t_1)/\nu, \tag{3.17}$$

$$y_1(t,\tau) = \frac{b \sin t}{y_0^2(\tau)-1} . \tag{3.18}$$

Since at the beginning of region $\bar{A}$ $(x,t) = (-2,\bar{t}_0)$ we derive from (3.16)-(3.18)

$$t_1 = \bar{t}_0 - (\tfrac{3}{2} - \log 2)\nu + \tfrac{b}{2} \sin \bar{t}_0 + o(\nu^{-1}). \tag{3.19}$$

Region $\bar{B}$

For region $\bar{B}$ we have

$$x = -1 - Z_0(t)\nu^{-1/2} + o(\nu^{-1/2}), \tag{3.20}$$

$$Z_0(t) = \sqrt{-b \sin t + t_1 - t} .$$

At region $\bar{B}$ the intersection with strip $x' = 1 + o(\nu^{-1/2})$ takes place at $t = \bar{t}_1$ satisfying

(3.21) $$t_1 - \bar{t}_1 = b \sin \bar{t}.$$

At this point the solution jumps to x = 2 under the condition

(3.22) $$\cos \bar{t}_1 > - 1/b.$$

At x = 2 the solution has been before at time $t = \bar{t}_{-1}$ according to (3.3) this was for

(3.23) $$\bar{t}_{-1} = t_0 - (\tfrac{3}{2} - \log 2)\nu + \tfrac{b}{2} \sin \bar{t}_{-1} + o(\nu^{-1}).$$

<u>Periodicity conditions</u>

We consider periodic solutions with period T being a multiple of 2π which intersect the line twice in a period. Such solution satisfies

(3.24) $$\bar{t}_1 - \bar{t}_{-1} = 2\pi m.$$

Let $2\delta(\nu)$ be the difference between the period $T_0(\nu)$ of the autonomous equation and the period T of the special solution, then

(3.25) $$2\delta = T_0 - T = (3-2 \log 2)\nu - 2\pi m + O(\nu^{-1/3}).$$

The system of equations (3.14), (3.19), (3.21), (3.23) and (3.24) can be reduced to

(3.26a) $$3b(\sin \bar{t}_0 - \sin \bar{t}_1) = -4\delta,$$

(3.26b) $$b(\sin \bar{t}_0 + \sin \bar{t}_1) = -4(\bar{t}_1 - \bar{t}_0) + 4\pi m.$$

It turns out that the following change is suitable for the calculations

(3.27) $$\begin{aligned} \bar{t}_{-1} &= 2k_{-1}\pi + v_{-1} \\ \bar{t}_0 &= (2k_0+1)\pi + v_0 \\ \bar{t}_1 &= 2k_1\pi + v_1 \end{aligned}$$

with $-\pi < v_i \leq \pi$, $i = -1,0,1$. In view of the periodicity we have $v_{-1} = v_1$. For $b \leq 1$ equations (3.14) and (3.21) have a unique solution; for $b > 1$ we have to select the smallest root. In terms of v_i the following condition has to be satisfied

(3.28) $$v_i + b \sin v_i > \sqrt{b^2-1} - \arccos(\tfrac{1}{b}) - \pi, \qquad i = 0,1.$$

Conditions (3.15) and (3.22) transform into

(3.29) $$\cos v_i > -1/b, \qquad i = 0,1.$$

<u>The case m odd</u>

For

(3.30a) $$2k_1 - (2k_0+1) = (2k_0+1) - 2k_{-1} = m$$

(3.30b) $$v_0 = v_1$$

Equation (3.26b) is satisfied. Substitution in equation (3.26a) gives

(3.31) $$v_0 = v_1 = \arcsin\left(\frac{2\delta}{3b}\right),$$

so another natural restriction of the parameters is

(3.32) $$\left|\frac{2\delta}{3b}\right| \leq 1.$$

Conditions (3.29) are satisfied by (3.31), while (3.28) reads

(3.33) $$\arcsin \frac{2\delta}{3b} + \frac{2}{3} b > \sqrt{b^2-1} - \arccos(\tfrac{1}{b}) - \pi.$$

In the b,ν-plane (3.32) and (3.33) determine the region, where a subharmonic solution with period $2\pi m$ with m odd may be expected, see figure 3.2. These are symmetric solutions satisfying $x(t) = -x(t-\frac{1}{2}T)$.

<u>The case m even</u>

If we set

(3.34a) $$2k_1 - (2k_0+1) = m - 1,$$

(3.34b) $\quad (2k_0+1) - 2k_{-1} = m + 1,$

the system (3.26) does not admit a solution of the type (3.30b). Besides the necessary condition (3.32) we find also by taking $v_{1\pi} = v_1 + \pi$ and applying the mean value theorem

$$\frac{\sin^2 v_0 - \sin^2 v_{1\pi}}{v_0 - v_{1\pi}} = \frac{16\delta}{3b^2}$$

the (solvability) condition

(3.35) $\quad \left|\frac{16\delta}{3b^2}\right| \leq 1.$

In figure 3.2 we also give the region where a numerical solution for (3.26) was found that satisfied (3.28), (3.29), (3.32) and (3.35).

Some remarks

The regions in the b,ν-plane corresponding with subharmonics of period $2\pi(2n-1)$ and $2\pi(2n+1)$ overlap. For a value of b and ν in the domain of overlap two different periodic solutions are possible depending on the initial values. The region corresponding with a subharmonic of period $4\pi n$ overlaps the two regions mentioned above in such a way that in a very narrow strip three subharmonics might exist. It is also possible to construct subharmonic solutions that intersect the line $x = 0$ $2q$ times $(q = 2,3,\ldots)$ in one period, $T = (3-2\log 2)q\nu + O(1)$. This would lead to a system of $2q$ equations of the type (2.6). Such system can easily be reduced to a system of q equations in case of symmetric solutions with $x(t) = -x(t-\frac{1}{2}T)$. Finally, we remark that it is also possible to give sufficient conditions for solving the system (2.6) with m even. These conditions read

(3.36) $\quad 1 - \theta > \frac{4}{6}(\arccos\sqrt{\theta} + \frac{\pi}{2}) \quad \text{or} \quad 1 - \theta > \frac{4}{b}\arcsin\sqrt{\theta},\ \theta = 2\delta/(3b).$

In Littlewood's study [14] the amplitude b of the forcing term is of order $O(\nu)$. This leads to a same structure of subharmonic solutions with period $T = 2\pi(2n\pm1)$ as found for b sufficiently large but independent of ν. Littlewood states that for $b = \beta\nu$ with $\beta > 2/3$ only stable solutions of period 2π are found. Moreover, he signalized a what he called dipping phenomenon: the solution dips one or more times below the line $x = 1$ before jumping to the value $x = -2$ (a similar phenomenon may occur at $x = -1$).

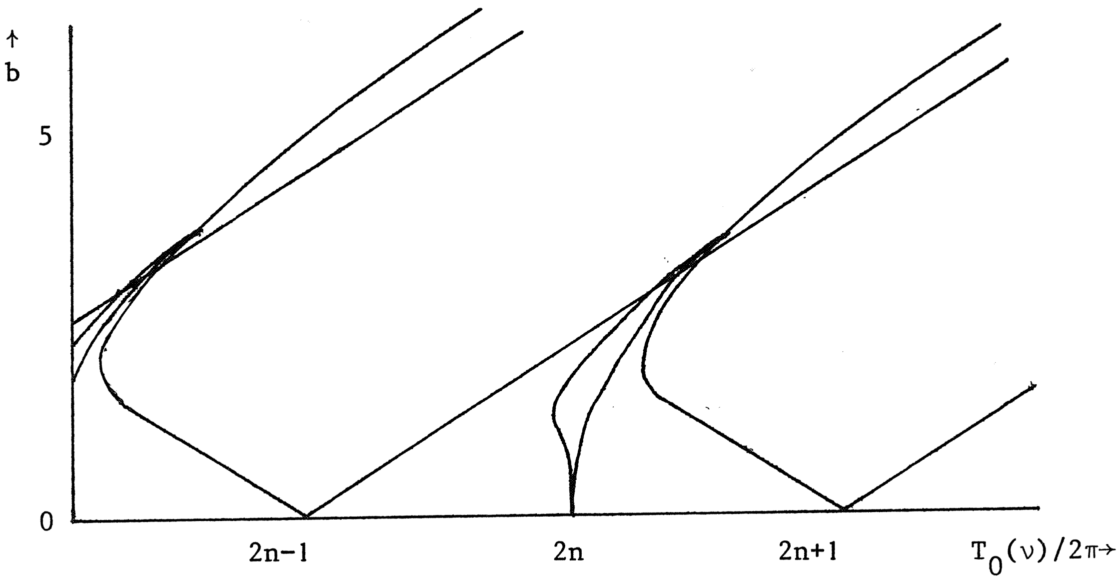

Fig. 3.2 Regions in the b,ν-plane with subharmonic solutions

4. WEAKLY COUPLED RELAXATION OSCILLATIONS

In this section we will present results on *coupled relaxation oscillations*. For the proofs of lemma's and theorems and for generalizations and further applications we refer to [10] and [11]. We consider a system of n coupled Van der Pol oscillators

$$\varepsilon \frac{d^2u_i}{dt^2} + (u_i^2-1)\frac{du_i}{dt} + u_i = \delta \sum_{j=1}^{n} h_{ij}(u_j), \qquad i = 1,2,\ldots,n, \tag{4.1}$$

where h_{ij} is a Lipschitz continuous function ($h_{ii}=0$) and $0 < \varepsilon, \delta \ll 1$. This system can be transformed into

$$\varepsilon \frac{du_i}{dt} = v_i - \frac{1}{3}u_i^3 + u_i \tag{4.2a}$$

$$\frac{dv_i}{dt} = -u_i + \delta \sum_{j=1}^{n} h_{ij}(u_j), \qquad i = 1,2,\ldots,n. \tag{4.2b}$$

We also consider the degenerated system ($\varepsilon=0$)

$$0 = v_{0i} - \frac{1}{3}u_{0i}^3 + u_{0i} \tag{4.3a}$$

$$\frac{dv_{0i}}{dt} = -u_{0i} + \delta \sum_{j=1}^{n} h_{ij}(u_{0j}), \qquad i = 1,2,\ldots,n. \tag{4.3b}$$

We introduce *formal discontinuous limit solutions* $(u_{0j}(t), v_{0j}(t))$ that satisfy (4.3) on regular arcs in the phase space with $|u_{0j}| > 1$. These arcs are connected by lines with v_{0j} and u_{0j} $(j \neq i)$ constant and with u_{0i} varying from ± 1 to ∓ 2, denoting instantaneous jumps in u_{0i}. In the sequel it is assumed that at the end of a regular arc only one of the variables u_j equals ± 1. If such a sequence of connected arcs and lines forms a closed trajectory $Z_0^{(n)}$, then we have constructed a *formal discontinuous periodic limit solution*; its period $T_0^{(n)}$ is found by integration of (4.3b) over the regular arcs. For $n = 1$ we have the autonomous Van der Pol equation with $Z_0^{(1)}$ as sketched in figure 4.1. We denote the discontinuous periodic limit solution by

(4.4) $$u_{01}(t) = x_0(t), \quad v_{01}(t) = y_0(t).$$

Its period satisfies $T_0^{(1)} = 3 - 2 \log 2$.

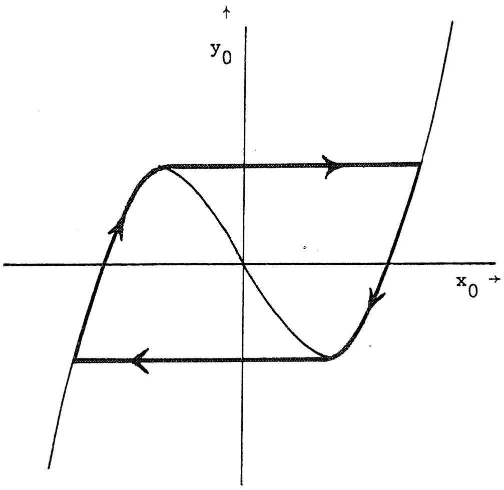

Fig. 4.1. The closed trajectory $Z_0^{(1)}$

Let us consider an (n-1)-dimensional surface P in $\mathbb{R}^{2n}$ satisfying (4.3a), $i = 1,\dots,n$, being transversal to $Z_0^{(n)}$ at a point q of a regular arc. A formal discontinuous limit solution starting at P near q will return in a neighbourhood of q. In this way a mapping Q: P → P is defined. MISHCHENKO [15] proved the following theorem.

THEOREM 4.1. *Let the system* (4.2) *with degeneration* (4.3) *have a discontinuous periodic limit solution with closed trajectory* $Z_0^{(n)}$ *and let at a point* q *on a regular arc have a mapping* Q: P → P *be defined as above. If* Q *and its linearization at* q *are contracting, then the system* (4.2) *has for* ε *suffi-*

ciently small a periodic solution with period $T_\varepsilon^{(n)}$ *and closed trajectory* $Z_\varepsilon^{(n)}: T_\varepsilon^{(n)} = T_0^{(n)} + O(\varepsilon^{2/3})$ *and* $Z_\varepsilon^{(n)} \to Z_0^{(n)}$ *as* $\varepsilon \to 0$.

From now on we focus our attention to the construction of $Z_0^{(n)}$. For δ sufficiently small we may write for the i-th component as a function of time

$$(4.5a) \qquad u_{0i}(t) = x_0(\phi_i(t)),$$

$$(4.5b) \qquad v_{0i}(t) = y_0(\phi_i(t)),$$

where (x_0, y_0) is the discontinuous approximation of the autonomous Van der Pol equation. That is the i-th component of (4.3) runs the closed trajectory $Z_0^{(1)}$ of the autonomous Van der Pol equation in the limit $\varepsilon \to 0$. Substitution in (4.3b) gives

$$(4.6) \qquad \frac{d\phi_i}{dt} = 1 - \delta \sum_{j=1}^{n} h_{ij}[x_0(\phi_j)]/x_0(\phi_i), \qquad i = 1,2,\dots,n.$$

The value of ϕ_i may be taken modulo $T_0^{(1)}$. Thus the problem is reduced to a system of n differential equations with function values on a n-dimensional torus. Let us set the discontinuities of $x_0(\phi)$ at $\phi = 0$ and $\phi = T_0^{(1)}/2$. We call a point $\alpha \in \mathbb{R}^n$ *regular* if the functions $u_{0i} = x_0(\alpha_i + t)$ are continuous in $t = 0$ and if they are discontinuous one at a time for $t > 0$.

<u>LEMMA 4.1</u>. *Let* $\phi(0) = \alpha$ *be regular. Then equation* (4.6) *has a unique solution* $\phi(t)$. *Moreover, for* t *bounded (independent of* ε)

$$(4.7) \qquad \phi_i(t) = \alpha_i + t - \delta \sum_{j=1}^{n} \int_0^t \frac{h_{ij}[x_0(\alpha_j+\tau)]}{x_0(\alpha_i+\tau)} \, d\tau + O(\delta^2).$$

Let V be a (n-1)-dimensional plane orthogonal to $e = (1,1,\dots,1)$ in $\mathbb{R}^n$. Let $\phi(0) \in V$ and let $T^*(\phi(0))$ be the time at which $\phi(t)$ returns in V. We consider the mapping Q_V from V into V defined by

$$(4.8) \qquad \phi(0) \mapsto \phi(T^*(\phi(0))), \qquad T^*(\phi(0)) = T_0^{(1)} + \tau^*(\phi(0)),$$

see figure 4.2 for n = 2. Clearly, we must have $\tau^*(\phi(0)) = O(\delta)$. The first order approximation of Q_V with respect to δ reads

$$(4.9) \qquad Q_V^{(0)}(\alpha) = \alpha + \delta G(\alpha) + \tau^*(\alpha)e,$$

where

$$G_i(\alpha) = -\sum_{j=1}^{n}\int_0^{T_0^{(1)}} \frac{h_{ij}[x_0(\tau)]}{x_0(\alpha_i-\alpha_j+\tau)}\,d\tau,$$

$$\tau^*(\alpha) = -\frac{\delta}{n}\sum_{i=1}^{n} G_i(\alpha).$$

Let this mapping have a fixed point $\tilde{\alpha}$: $Q_V^{(0)}(\tilde{\alpha}) = \tilde{\alpha}$.

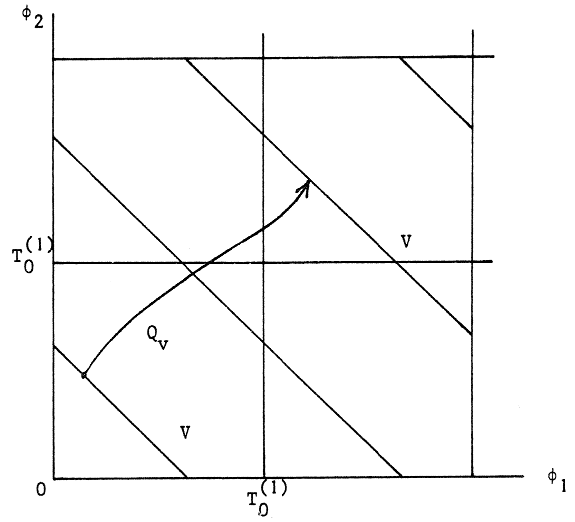

Fig. 4.2. The mapping Q_V in the phase-space

LEMMA 4.2. *The mapping* Q_V *and its linearization are contracting for* δ *sufficiently small, if the eigenvalues of the Jacobian of* $Q_V^{(0)}$ *are lying within the unit circle.*

For the problem (4.2) Mishchenko's theorem can be reformulated as follows.

THEOREM 4.2. *Let the system* (4.2) *have a formal discontinuous periodic limit solution satisfying* (4.3) *on the regular arcs and with jumps as prescribed. Let a point* q *on a regular arc be fixed point of a mapping* Q_V. *If the eigenvalues of the Jacobian of* $Q_V^{(0)}$ *are within the unit circle, then* (4.2) *has a periodic solution with period* $T_{\varepsilon,\delta}^{(n)}$ *and closed trajectory* $Z_{\varepsilon,\delta}^{(n)}$ *satisfying*

$$T_{\varepsilon,\delta}^{(n)} = T_0^{(1)} + \tau^*(\tilde{\alpha}) + O(\delta^2) + O(\varepsilon^{2/2})$$

and

$$Z^{(n)}_{\varepsilon,\delta} \to Z^{(n)}_0 \quad \text{as} \quad \varepsilon,\delta \to 0.$$

It is noted that $\tau^*(\tilde{\alpha})$ is usually different from zero, which means that the period of the total system differs $O(\delta)$ from the period of the individual oscillators in decoupled state.

EXAMPLE 4.1. A system of Van der Pol oscillators on a circle with each oscillator only coupled with its direct neighbours may have the form

$$(4.10a) \qquad \varepsilon \frac{d^2u_1}{dt^2} + (u_1^2-1)\frac{du_1}{dt} + u_1 = \delta(u_n+u_1),$$

$$(4.10b) \qquad \varepsilon \frac{d^2u_i}{dt^2} + (u_i^2-1)\frac{du_i}{dt} + u_i = \delta(u_{i-1}+u_{i+1}), \qquad i = 2,\dots,n-1$$

$$(4.10c) \qquad \varepsilon \frac{d^2u_n}{dt^2} + (u_n^2-1)\frac{du_n}{dt} + u_n = \delta(u_{n-1}+u_1).$$

Considering the phase ϕ as a function of time and (discretized) position: $\phi = \phi(t,k\theta)$ with $\theta = 2\pi/n$, $k = 0,1,\dots,n-1$, we may find wave-type solutions satisfying

$$(4.11) \qquad n\{\phi(t,k+1)\theta)-\phi(t,k\theta)\} = mT_0^{(1)}, \qquad k = 0,1,\dots,n-1,$$

with the circumference of the circle being m times the wave length. The conditions for theorem 4.2 are satisfied if $n \times m$ = odd (one oscillator jumps at a time) and if the eigenvalues of the Jacobian of $Q_v^{(0)}$ are within the unit circle, that is if

$$G'(\mu) - G'(-\mu) < 0$$

with

$$G(\mu) = -\int_0^{T_0^{(1)}} \frac{x_0(\tau)}{x_0(\tau+\mu)}\, d\tau, \qquad \mu = \frac{m}{n} T_0^{(1)}.$$

The results for this system of oscillators on a circle strongly resembles the behaviour of a model chemical reaction with diffusion taking place in a ring-shaped domain. AUCHMUTY and NICOLIS [1] analyzed this model reaction, first formulated by Prigogine, and found wave-type solutions similar to (4.11). Investigations on this model chemical reaction led to a better understanding of periodic phenomena in biochemistry and other areas of biology.

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